Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Topics in Discrete Probability

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Max. points : 40. **Time Limit : 3 hours.**

Total points are 50. Answer as many questions as you can.

Give necessary justifications and explanations for all your arguments. Mention clearly results from the class or assignments.

1. Let $C(G)$ be the size of the largest clique (i.e., a complete subgraph) in a graph G. Let $G(n,p)$ denote the Erdös-Rényi graph with n vertices and the probability of an edge being present is p. Find the threshold function in p for the property $\{C(G) \geq 5\}$. (**10**)

2. Let $d \geq 1$ and consider bond percolation on $\mathbb{Z}^d$ with probability $p \geq 0$.

   (a) Let $x_1, \ldots, x_k \in \mathbb{Z}^d$ and $\tau_p(x_1, \ldots, x_n)$ to be the probability that $x_1, \ldots, x_k$ belong to the same cluster. Show that $\tau_p(x_1, \ldots, x_n)$ is continuous as a function of p for all $x_1, \ldots, x_k \in \mathbb{Z}^d$. (**5**)

   (b) For $S \subset \mathbb{Z}^d$, define $\phi_p(S) := p \sum_{x \in S, y \notin S} \mathbb{P}_p(o \overset{S}{\leftrightarrow} x)$. Show that if there exists a finite S such that $O \in S$ and $\phi_p(S) < 1$, then there exists a $c_p > 0$ such that $\mathbb{P}_p(O \leftrightarrow \partial \Lambda_n) \leq e^{-c_p n}$. (**5**)

   (c) Let $p \in (0,1)$. Let G_n denote the restriction of bond percolation to Λ_n and $K_{k,n}$ denote the number of components of size k in G_n. Show that $\mathbb{E}(K_{k,n}) = \Omega(n)$ and further for any $\epsilon > 0$, show that
   $$\frac{K_{k,n} - \mathbb{E}(K_{k,n})}{n^{1/2+\epsilon}} \xrightarrow{P} 0. \qquad (\mathbf{6})$$

3. A *cut* in a graph is a set of edges of the form $\{(x,y) : x \in A, y \notin A\}$ for some proper non-empty set A of G. Show that for every finite network G, the linear span of $\{\sum_{e \in \Pi} c(e)\chi^e : \Pi \text{ is a cut}\}$ is the star space of G. (**4**)

4. Suppose that L is a function of subsets of $\mathbb{R}^d$ that is monotone in the sense that $L(\{x_1,\dots,x_n\}) \leq L(\{x_1,\dots,x_{n+1}\})$ and suppose further that there are non-negative weight functions $\alpha_i(x)$ for which for all $x = \{x_1,\dots,x_n\}$ and $y = \{y_1,\dots,y_n\}$, the set functional L satisfies $L(\{x_1,\dots,x_n\}) \leq L(\{y_1,\dots,y_n\}) + \sum_{i=1}^n \alpha_i(x)1(x_i \neq y_i)$. Let the weight functions $\alpha_i(x)$ satisfy the uniform bound $\sum_i \alpha_i(x)^2 \leq c^2$. Show that if $X_1,\dots,X_n$ are i.i.d. uniform random variables on $[0,1]^d$, we have that

$$\mathbb{P}(|L(X_1,\dots,X_n) - M_n| \geq t) \leq 4e^{-t^2/4c^2},$$

where M_n is a median of $L(X_1,\dots,X_n)$. (**5**)
See below for statement of Talagrand's inequality.

5. Let L_n be the length of a longest increasing subsequence of a uniform random permutation on $[n]$ and M_n be a median of L_n. Show that $|L_n - M_n| = o(\sqrt{n})$ and $\mathbb{VAR}(L_n) = o(\sqrt{n})$. (**5**)

6. Suppose we throw m balls into n bins independently, uniformly at random and $Z_{n,m}$ be the number of empty bins. Show that

$$\mathbb{P}(|Z_{n,m} - n(1-\frac{1}{n})^m| \geq b\sqrt{m}) \leq 2e^{-b^2/2}. \qquad (\mathbf{5})$$

7. Consider the lattice graph $\mathbb{Z}^d, d \geq 1$ with unit conductances on all edges. Define G_n to be the subnetwork induced by restriction to $\mathbb{Z}^d \cap [-n,n]^d$. Show that there exists $C_d > 0$ such that for all $x, y \in \mathbb{Z}^d \cap [-n,n]^d$, we have that

$$\mathcal{R}(x \leftrightarrow y; G_n) \geq \begin{cases} C_d|x-y| \ ; \ d = 1, \\ C_d \log|x-y| \ ; \ d = 2, \\ C_d \ ; \ d \geq 3. \end{cases} \qquad (\mathbf{5})$$

Talagrand's convex distance and concentration Inequality : For $x = (x_1,\dots,x_n) \in (\mathbb{R}^d)^n$ and $A \subset (\mathbb{R}^d)^n$, define

$$d_T(x,A) = \sup\{z_\alpha : z_\alpha = \inf_{y\in A} \sum_{1\leq i\leq n} \alpha_i(x)1(x_i \neq y_i) \& \sum_i \alpha_i(x)^2 \leq 1\}.$$

Talagrand's inequality states that $P(A)P(d_T(x,A) > t) \leq e^{-t^2/4}$ where P is a product measure on $(\mathbb{R}^d)^n$ and $t > 0$.